

On the structure of generators for non-Markovian Master Equations

Andrzej Kossakowski

Institute of Physics

Nicholaus Copernicus University

87-100 Torun, Poland

kossak@fizyka.umk.pl

and

Rolando Rebolledo *

Laboratorio de Análisis Estocástico

Facultad de Matemáticas

Pontificia Universidad Católica de Chile

Casilla 306, Santiago 22, Chile

rrebolle@puc.cl

Abstract

Complete characterization of complete positivity preserving non-Markovian master equations is presented.

1 Introduction

The study of time evolution of quantum open systems plays an important role in quantum information. The interaction between the system and its environment leads to phenomena of decoherence and dissipation [1, 2, 3].

The Nakajima-Zwanzig projector operator method [4, 5] makes possible to derive an exact equation for the reduced density matrix from the von Neumann equation of the composed system. The resulting non-Markovian master equation is mostly of formal interest since its solution cannot be written down explicitly, in closed form. In contrast, when the Markovian

*Research supported by the Bicentennial Program PBCT-ACT13, Chile and the Program of Visiting Scholars of the Pontificia Universidad Católica de Chile

approximation is used, that is, when memory effects are neglected, the resulting Markovian master equation [6, 7] has a simpler form and complete positivity is preserved during the evolution [8].

A challenge for the non-Markovian theory of open quantum systems is to obtain a characterization of a class of evolutions which preserves complete positivity and captures reservoir memory effects at the same time.

A variety of non-Markovian master equations have been proposed (cf. [2, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]). However, the complete positivity of the resulting time evolution remains an important problem [32, 33]. In the present paper the structure of non-Markovian master equations preserving complete positivity is given.

2 Notations and preliminaries

To avoid technical difficulties connected with infinite dimensions, we restrict ourselves to the d -dimensional Hilbert space \mathbb{C}^d of complex vectors with the scalar product $\langle \cdot, \cdot \rangle$ and elements denoted e, x, y, z, \dots . The extension to infinite dimensional Hilbert spaces is currently being investigated and will appear in a forthcoming paper.

The C^* -algebra of linear operators on \mathbb{C}^d will be denoted $M_d = M_d(\mathbb{C})$, or $\mathfrak{L}(\mathbb{C}^d)$. Elements of M_d will be denoted by a, b, c, \dots and its unit by $\mathbf{1}$. M_d is also a Hilbert space under the scalar product $\langle a, b \rangle = \text{tr}(a^*b)$.

The C^* -algebra of linear maps from M_d into M_d will be denoted by $\mathfrak{L}(M_d)$, its elements are denoted by capital letters A, B, C, \dots and the identity map in $\mathfrak{L}(M_d)$ by I . The conjugation (duality) $\cdot^\#$ in $\mathfrak{L}(M_d)$ is defined by the relation:

$$\langle A^\#a, b \rangle = \langle a, Ab \rangle, \quad (2.1)$$

for all $a, b \in M_d$.

This operation endows the following property: the relations

$$A\mathbf{1} = \mathbf{1}, \quad L\mathbf{1} = 0, \quad (2.2)$$

and

$$\text{tr}(A^\#a) = \text{tr}(a), \quad \text{tr}(L^\#a) = 0, \quad (2.3)$$

are equivalent.

The cone of all completely positive maps on M_d will be denoted by $\mathfrak{L}^+(M_d)$. Within this paper we make an intensive use of Laplace Transform Theory. In general we reserve the symbol $\hat{\cdot}$ and the variable p for

Laplace transforms. In general we will consider scalar functions $f(t)$ *positive* and integrable on $[0, \infty]$ and denote its Laplace transform by $\hat{f}(p) = \int_0^\infty dt \exp(-pt)f(t)$. On the other hand, measurable $\mathfrak{L}(M_d)$ -valued functions will be written like $A_t \in \mathfrak{L}(M_d)$, $t \geq 0$. For such a function the Laplace transform exists and will be denoted \hat{A}_p .

3 Non-Markovian master equations

The reduced dynamics can be studied equivalently in the Schrödinger or the Heisenberg pictures. Suppose that $A_t : M_d \rightarrow M_d$ describes the reduced dynamics in the Heisenberg picture, then it should satisfy the following conditions: $A_t \in \mathfrak{L}^+(M_d)$, $A_t \mathbf{1} = \mathbf{1}$, for all $t \geq 0$, and $A_0 = \lim_{t \rightarrow 0} A_t = I$. In the Schrödinger picture these relations are given in terms of $A_t^\#$, $t \geq 0$.

In the present paper the reduced dynamics is investigated under the assumption that A_t is the solution of a non-Markovian master equation of the form:

$$\frac{dA_t}{dt} = LA_t + \int_0^t ds L_{t-s} A_s, \quad (3.1)$$

with the initial condition $A_0 = I$.

The normalization condition $A_t \mathbf{1} = \mathbf{1}$ implies the equality

$$L_t \mathbf{1} = 0, \quad L \mathbf{1} = 0. \quad (3.2)$$

Let us observe that L can be formally absorbed in L_t by the transformation

$$L_t \mapsto L'_t = L_t + 2\delta(t) L. \quad (3.3)$$

In the present paper the equation

$$\frac{dA_t}{dt} = \int_0^t ds L_{t-s} A_s. \quad (3.4)$$

will be considered.

The map L_t will be referred as the *generator of the Master Equation*. One of the fundamental problems of non-Markovian master equations is to find conditions on L_t that ensure that the time evolution resulting from (3.4) is completely positive. The result of our previous paper [32] can be reformulated in the following manner:

Theorem 1 *Let be given a family of maps $Z_t \in \mathfrak{L}(M_d)$, ($t \geq 0$), such that*

$$L_t = B_t - Z_t, \quad (3.5)$$

where $B_t \in \mathfrak{L}^+(M_d)$ for all $t \geq 0$, and

$$L_t \mathbf{1} = B_t \mathbf{1} - Z_t \mathbf{1} = 0. \quad (3.6)$$

Then the time evolution A_t resulting from (3.4) is completely positive if the solution of the normalization equation

$$\frac{d}{dt} N_t = - \int_0^t ds Z_{t-s} N_s; \quad N_0 = I, \quad (3.7)$$

is completely positive.

This version of the theorem leads to the difficult question of obtaining conditions on Z_t guaranteeing that N_t is completely positive for any $t \geq 0$. The case $Z_t = \frac{1}{2}(c_t a + a c_t^*)$ and $c_t c_t^* = c_t^* c_t$ has been investigated in [33]. The construction of Z_t for which N_t is completely positive is given in the following Theorem.

Theorem 2 *Suppose the solution N_t of the normalization equation is completely positive and given in the following form*

$$N_t = I - \int_0^t ds F_s, \quad (3.8)$$

where $t \mapsto F_t$ is an integrable $\mathfrak{L}(M_d)$ -valued function.

Then Z_t is the solution of the integral equation

$$\int_0^t ds N_{t-s} Z_s = F_t. \quad (3.9)$$

Proof. Taking the Laplace transform of (3.7) yields

$$\widehat{N}_p = (pI + \widehat{Z}_p)^{-1} \quad (3.10)$$

By hypothesis

$$\widehat{N}_p = \frac{1}{p}(I - \widehat{F}_p), \quad (3.11)$$

so that comparing with (3.10) we get

$$\widehat{Z}_p = p \widehat{F}_p (I - \widehat{F}_p)^{-1}. \quad (3.12)$$

It is a straightforward computation to verify that the map Z_t is the solution of (3.9). \square

The Theorem 1 has been recently generalized by Breuer and Vaccini [34].

Theorem 3 (Breuer-Vaccini) *Suppose that the generator L_t is given in the form*

$$L_t = B_t - Z_t, \quad t \geq 0, \quad (3.13)$$

where

$$L_t \mathbf{1} = B_t \mathbf{1} - Z_t \mathbf{1} = 0, \quad (3.14)$$

the solution A_t , $t \geq 0$, of (3.4) is completely positive if the solution N_t , $t \geq 0$, of the normalization equations (3.7) is complete positive and the map

$$\int_0^t ds N_{t-s} B_s \quad (3.15)$$

is completely positive for all $t \geq 0$.

The application of Theorems 2 and 3 is illustrated as follows.

Example 1 Suppose that N_t is of the form

$$N_t = \left(1 - \int_0^t ds f(s)\right) I, \quad (3.16)$$

where $f(s)$ is a positive measurable scalar function, such that

$$\int_0^\infty ds f(s) \leq 1, \quad (3.17)$$

this means that one has $F_t = f(t)I$. It follows from (3.9) that

$$Z_t = \kappa(t)I, \quad (3.18)$$

where $\kappa(t)$ is given in terms of its Laplace transform

$$\widehat{\kappa}(p) = \frac{p \widehat{f}(p)}{1 - \widehat{f}(p)} \quad (3.19)$$

Let B_t , $t \geq 0$, be given in the form

$$B_t = \kappa(t)B, \quad (3.20)$$

where B is completely positive and normalized ($B1 = 1$) map on M_d . Using (3.16), (3.19) and (3.20) one finds

$$\int_0^t ds N_{t-s} B_s = f(t)B, \quad (3.21)$$

i.e., the condition (3.15) is satisfied.

Corollary 1 *The family L_t , $t \geq 0$, of maps:*

$$L_t = \kappa(t)(B - I), \quad (3.22)$$

where B is completely positive and normalized map on M_d , is the generator of a non-Markovian master equation provided $\kappa(t)$ is the solution of the integral equation

$$\int_0^t ds g(t-s) \kappa(s) = f(t), \quad (3.23)$$

where

$$g(t) = 1 - \int_0^t ds f(s), \quad (3.24)$$

and $f(t)$ is a positive measurable function satisfying (3.17).

The above result can be easily generalized. Let B be as above, and F_t defined by (3.8) is completely positive for all $t \geq 0$, then

$$\mathcal{L}_t = Z_t(B - I), \quad (3.25)$$

where Z_t is given by (3.9), is the generator of non-Markovian master equation.

The reduced dynamics A_t , $t \geq 0$ is characterized as a family of completely positive and normalized maps which is the solution of (3.4) under the initial condition $A_0 = I$. The reduced dynamics can also be characterized as follows

Theorem 4 *Let A_t , $t \geq 0$ be the reduced dynamics, then A_t has the representation*

$$A_t = I + \int_0^t ds G_s, \quad (3.26)$$

where

$$G_s(\mathbf{1}) = 0. \quad (3.27)$$

Proof. The reduced dynamics is defined as follows

$$A_t a = \text{tr}_{\mathcal{H}} \omega[e^{tL}(a \otimes \mathbf{1}_{\mathcal{H}})], \quad (3.28)$$

where e^{tL} is completely positive and normalized semigroup on $M_d \otimes B(\mathcal{H})$, ω is a fixed normal state on $B(\mathcal{H})$ and $\mathbf{1}_{\mathcal{H}}$ is the identity map on \mathcal{H} . It follows from (3.28) that (3.26) holds with

$$G_t a = \text{tr}_{\mathcal{H}} \omega [L e^{tL} (a \otimes \mathbf{1}_{\mathcal{H}})]. \quad (3.29)$$

Theorem 5 *Let A_t , $t \geq 0$ be a family of completely positive and normalized maps satisfying the equation*

$$\frac{dA_t}{dt} = \int_0^t ds L_{t-s} A_s, \quad A_0 = I \quad (3.30)$$

i.e.,

$$\hat{A}_p = \frac{1}{\mathbf{1} - \hat{L}_p}, \quad (3.31)$$

then the relations

$$A_t = \mathbf{1} + \int_0^t ds G_s \quad (3.32)$$

where

$$G_s(1) = 0, \quad (3.33)$$

and

$$\hat{L}_p = \frac{p \hat{G}_p}{\mathbf{1} + \hat{G}_p} \quad (3.34)$$

are equivalent.

Proof. From (3.32) one obtains

$$\hat{A}_p = \frac{1}{p} (\mathbf{1} + \hat{G}_p). \quad (3.35)$$

From (3.31) and (3.35) it follows that relation (3.34) hold. On the other hand inserting (3.34) into (3.31) one finds (3.32).

Example 2 Let $A_t = e^{tL}$, $t \geq 0$ be a completely positive semigroup. One can write A_t in the form

$$A_t = \mathbf{1} + \int_0^t ds G_s \quad (3.36)$$

where

$$G_t = L e^{tL}. \quad (3.37)$$

Using the relation (3.34) and (3.37) one finds

$$\widehat{L}_p = L, \quad (3.38)$$

i.e.

$$L_t = 2\delta(t)L \quad (3.39)$$

and (3.30) takes the form

$$\frac{dA_t}{dt} = LA_t. \quad (3.40)$$

Example 3 Let L_1, \dots, L_n be generators of completely positive semi-groups and $x_1, \dots, x_n \geq 0$,

$$\sum_{j=1}^n x_j = 1. \quad (3.41)$$

Let A_t , $t \geq 0$ be the family of completely positive normalized maps defined as follows

$$A_t = \sum_{j=1}^n x_j e^{tL_j}. \quad (3.42)$$

A_t can be rewritten in the form

$$A_t = \mathbf{1} + \int_0^t ds G_s, \quad (3.43)$$

where

$$G_t = \sum_{j=1}^n x_j L_j e^{tL_j}. \quad (3.44)$$

Taking the Laplace transform of G_t and using (3.34) one finds

$$\widehat{L}_p = \frac{p \sum_{j=1}^n x_j L_j (p - L_j)^{-1}}{\mathbf{1} + \sum_{j=1}^n x_j (p - L_j)^{-1}}. \quad (3.45)$$

Suppose that $[L_i, L_j] = 0$, $i, j = 1, \dots, n$ and consider the special cases $n = 2$ and $n = 3$.

For $n = 2$ it follows from (3.45) that

$$\widehat{L}_p = x_1 L_1 + x_2 L_2 + \frac{x_1 x_2 (L_1 - L_2)^2}{p - (x_1 L_2 + x_2 L_1)}. \quad (3.46)$$

In the case $n = 3$ one obtains

$$\widehat{L}_p = L + \frac{\widehat{B}_p}{\widehat{C}_p}, \quad (3.47)$$

where

$$L = x_1 L_1 + x_2 L_2 + x_3 L_3, \quad (3.48)$$

$$\begin{aligned} \widehat{B}_p &= p(x_1 L_1^2 + x_2 L_2^2 + x_3 L_3^2 - L^2) + L_1 L_2 L_3 \\ &\quad - L(x_1 L_2 L_3 + x_2 L_1 L_3 + x_3 L_1 L_2), \end{aligned} \quad (3.49)$$

$$\widehat{C}_p = p^2 - p(L_1 + L_2 + L_3 - L) + x_1 L_2 L_3 + x_2 L_1 L_3 + x_3 L_1 L_2. \quad (3.50)$$

Example 4 Let L_1, \dots, L_n be generators of completely positive semi-groups and $x_1(t), \dots, x_n(t)$ be nonnegative functions such that

$$\sum_{j=1}^n \int_0^t dx_j(s) \leq 1. \quad (3.51)$$

Let us consider the family A_t , $t \geq 0$ of completely positive and normalized maps defined as follows

$$A_t = \mathbf{1} \left(1 - \sum_{j=1}^n \int_0^t dx_j(s) \right) + \sum_{j=1}^n \int_0^t dx_j(s) e^{sL_j}. \quad (3.52)$$

The formula (3.52) can be rewritten in the form

$$A_t = \mathbf{1} + \int_0^t ds G_s, \quad (3.53)$$

where

$$G_t = \sum_{j=1}^n x_j(t) (e^{tL_j} - \mathbf{1}). \quad (3.54)$$

The generator \widehat{L}_p is given by (3.34) and it cannot be written explicitly.

The examples 3, 4 and 5 show that, in general, it is rather difficult to write down the generator L_t explicitly. However, if the family A_t , $t \geq 0$ of completely positive maps can be represented in the form (3.32) then A_t is the solution of non-Markovian master equation (3.30).

Acknowledgements

A part of this work was presented at the Torun 40th Symposium on Mathematical Physics. This symposium considered a special session in honor of Professor Andrzej Kossakowski. R.R. wish to acknowledge the kind invitation and hospitality of the Organizing Committee of the Symposium, which allowed him to express his deep gratitude to Andrzej for all the time afforded to a very fruitful joint scientific collaboration and friendship.

A.K. acknowledges the warm hospitality during the stay in PUC and a partial support by the Polish Ministry of Science and Higher Education Grant No 3004/B/H03/2007/33.

References

- [1] Alicki, R.; Lendi, K. *Quantum Dynamical Semigroups and Applications*, Vol. 286 of *Lecture Notes in Phys.*; Springer-Verlag, 1987.
- [2] Breuer, H.-P.; Petruccione, F. *The theory of open quantum systems*; Oxford University Press: New York, 2002.
- [3] Nielsen, M. A.; Chuang, I. L. *Quantum computation and quantum information*; Cambridge University Press: Cambridge, 2000.
- [4] Nakajima, S. *Prog. Theor. Phys.* **1958**, *20*, 948.
- [5] Zwanzig, R. *J. Chem. Phys.* **1960**, *33*, 1338.
- [6] Gorini, V.; Kossakowski, A.; Sudarshan, E. *J. Math. Phys.* **1976**, *17*, 821–825.
- [7] Lindblad, G. *Commun. Math. Phys.* **1976**, *48*, 119–130.
- [8] Kraus, K. *States, Effects and Operations, Fundamental notions of Quantum Theory*; Academics, Berlin, 1983.
- [9] Shibata, N.; Takahashi, Y. *J.Stat.Phys.* **1977**, *17*, 171.
- [10] Imamoglu, A. *Phys. Rev. A* **1994**, *50*, 3650.
- [11] Royer, A. *Phys. Rev. Lett.* **1996**, *77*, 3272.
- [12] Royer, A. *Phys. Lett. A* **2003**, *315*, 335.
- [13] Barnett, M.; Stenholm, S. *Phys. Rev. A* **2001**, *64*, 033808.

- [14] Breuer, H.-P.; Kappler, B.; Petruccione, F. *Ann. Physics* **2001**, *291*(1), 36–70.
- [15] Breuer, H. P.; Petruccione, F. *Phys. Rev. A* **2001**, *63*, 032102.
- [16] Breuer, H. P.; Kappler, B.; Petruccione, F. *Phys. Rev. A* **1999**, *59*, 1633.
- [17] Breuer, H. P. *Phys. Rev. A* **2004**, *69*, 022115.
- [18] Breuer, H. P. *Phys. Rev. A* **2004**, *70*, 012106.
- [19] Breuer, H.-P.; Gemmer, J.; Michel, M. *Phys. Rev. E (3)* **2006**, *73*(1), 016139, 13.
- [20] Maniscalco, S. *Phys. Rev. A* **2005**, *72*, 024103.
- [21] Maniscalco, S.; Petruccione, F. *Phys. Rev. A* **2006**, *73*, 012111.
- [22] Wilkie, J. *Phys. Rev. E* **2000**, *62*, 8808.
- [23] Wilkie, J. *J. Phys. A* **2001**, *114*, 7736.
- [24] Breuer, H. P. *Phys. Rev. A* **2007**, *75*, 022103.
- [25] Budini, A. *Phys. Rev. A* **2004**, *69*, 042107.
- [26] Budini, A. *Phys. Rev. A* **2006**, *74*, 053815.
- [27] Budini, A. quant-ph/0611222.
- [28] Shabani, A.; Lidar, D. *Phys. Rev. A* **2005**, *71*, 020101.
- [29] Lee, J.; Kim, I. A. D. M. A. H.; Kim, M. *Phys. Rev. A* **2004**, *70*, 024301.
- [30] Pülo, J.; Maniscalco, S. K. quant-ph/07064438, **2007**.
- [31] Kossakowski, A.; Rebolledo, R. *Open Syst. Inf. Dyn.* **2007**, *14*(3), 265–274.
- [32] Kossakowski, A.; Rebolledo, R. *Open Syst. Inf. Dyn.* **2008**, *15*(2), 135–141.
- [33] Breuer, H.-P.; Vaccini, B. *Phys. Rev. Lett* **101**, 140402 (2008).
- [34] Breuer H.-P., Vaccini, B., arXiv: quant-ph 09022318.